

Renormalization of the vacuum angle in quantum mechanics, Berry phase and continuous measurements

S. M. Apenko

Theory Department, Lebedev Physics Institute, Moscow, 119991, Russia*

(Dated: February 5, 2008)

The vacuum angle θ renormalization is studied for a toy model of a quantum particle moving around a ring, threaded by a magnetic flux θ . Different renormalization group (RG) procedures lead to the same generic RG flow diagram, similar to that of the quantum Hall effect. We argue that the renormalized value of the vacuum angle may be observed if the particle's position is measured with finite accuracy or coupled to additional slow variable, which can be viewed as a coordinate of a second (heavy) particle on the ring. In this case the renormalized θ appears as a magnetic flux this heavy particle sees, or the Berry phase, associated with its slow rotation.

PACS numbers: 11.10.Hi, 03.65.Vf, 05.10.Cc

In quantum field theories it is sometimes possible (e.g. in quantum chromodynamics (QCD) or non-linear σ models) to add the so-called topological term to the action and to consider the coefficient θ in front of this term, usually called vacuum or topological angle, as an additional parameter of the theory (see, e.g. [1]). Long ago it was suggested, that vacuum angle θ becomes scale dependent (as any other running coupling constant) if properly defined (non-perturbative) renormalization group (RG) transformation is introduced [2, 3], and flows to zero (mod 2π) in the infrared limit (see also [4] for some recent works). Hence one might expect that the observable low energy θ should vanish, possibly solving the strong CP problem in QCD (i.e. why we do not observe CP violation due to the θ -term while *a priori* there are no reasons to put $\theta = 0$ [5]). But such a renormalization is, in a sense, counter-intuitive, since θ more resembles some quantum number (and is related to a superselection rule) than usual coupling constant. Moreover, non-perturbative calculations, based on the sum rule approach [6] (see also [5] and references therein) have shown that CP violating effects actually depend on the bare θ , so that it is not clear what does the θ renormalization actually mean in QCD and how it may be observed.

Perhaps the most known example where such renormalization have proved to be important is the quantum Hall effect (QHE). In this case, described by a matrix non-linear σ model, the renormalized vacuum angle is in fact defined as the observable Hall conductivity, dependent on the sample's size or temperature (see e.g. [3, 7]).

Quite recently it became clear that charging effects in a single electron box (a metallic island coupled to the outside circuit by a tunnel junction), also described by a topological term, are closely related to the θ renormalization [8, 9]. This last model is equivalent to ordinary quantum mechanics of a particle (with friction in general case) on a ring threaded by a magnetic flux θ , which can

serve as the simplest zero dimensional toy model to study the θ renormalization in more detail.

It is possible to introduce a RG transformation in quantum mechanics, similar in spirit to decimation procedure in one dimensional classical spin models and related to continuous measurements theory, which leads to the θ renormalization of the required type [9], which manifest itself, as in QHE, as temperature dependence of a certain observable. Renormalization of θ is seen then to follow from the loss of information about the initial topological charge in the course of the RG transformation. The RG scheme of Ref. [9] is, however, somewhat artificial, since as a first step it introduces a lattice (like time slices in the Trotter decomposition, used e.g. in path integral Monte Carlo calculations [10]) to be removed in the end.

For this reason here we present a different RG approach, also inspired by an analogy between RG and continuous measurements, but with no lattice and at zero temperature. Now the renormalized θ appears as an effective magnetic flux seen by an additional slow variable (or Berry phase, related to its cyclic evolution, compare with [11]). The resulting RG flow diagram again has the typical QHE-like form with θ going to zero (mod 2π) in the infrared limit. Physical reasons for such behaviour are also discussed.

Consider a particle of mass m moving around a ring of unit radius threaded by a magnetic flux θ (in units $c = \hbar = e = 1$). The corresponding (euclidian) action at finite temperature may be written in terms of a planar unit vector $\mathbf{n}(\tau)$ ($\mathbf{n}^2 = 1$) which depends on a one-dimensional coordinate (euclidean time)

$$S_0[\mathbf{n}] = \frac{m}{2} \int_0^\beta \dot{\mathbf{n}}^2(\tau) d\tau - i \frac{\theta}{2\pi} \int_0^\beta \epsilon_{ab} n_a(\tau) \dot{n}_b(\tau) d\tau, \quad (1)$$

where ϵ_{ab} is the two dimensional antisymmetric tensor and β is the inverse temperature (we will assume $\beta \rightarrow \infty$ in what follows). Since $\mathbf{n}(0) = \mathbf{n}(\beta)$ the model is actually defined on a circle. The last term in (1) has the form $i\theta Q$ where Q is the topological charge which distinguishes inequivalent mappings $S^1 \rightarrow S^1$ and takes integer values

*Also at ITEP, Moscow, 117924, Russia; Electronic address: apenko@lpi.ru

(equal to a number of rotations the particle make in time β), making the theory periodic in θ .

The magnetic flux θ explicitly breaks T invariance, the most obvious T-violating effect being the non-zero persistent current in the ground state. This is the analog of the CP problem in QCD and now one may ask, how the dependence on θ can be removed. One possible answer is that the magnetic flux could be screened, if we allow the back reaction of the current on θ . This may be done by introducing an additional dynamical variable (axion), coupled to the topological charge density. Curiously, the model (1) with the axion have been introduced in a different context to describe a shunted Josephson junction [12].

Suppose now that we perform a continuous monitoring of the particle position (in euclidean time) with a finite accuracy. If a continuous quantum measurement results in a smooth slowly varying trajectory $\mathbf{n}_0(\tau)$ then the corresponding amplitude may be obtained through the restricted path integral [13]

$$U[\mathbf{n}_0] = \int D\mathbf{n}(\tau) \delta(\mathbf{n}^2(\tau) - 1) w[\mathbf{n}, \mathbf{n}_0] \exp(-S_0[\mathbf{n}]), \quad (2)$$

where the weight functional $w[\mathbf{n}, \mathbf{n}_0]$ is usually taken in a simple Gaussian form

$$w[\mathbf{n}, \mathbf{n}_0] = \exp\left(-\frac{\lambda}{2} \int_0^\beta [\mathbf{n}(\tau) - \mathbf{n}_0(\tau)]^2 d\tau\right) \quad (3)$$

and the constant λ determines the accuracy of the measurement.

Integration in Eq. (2) defines an effective action $U \sim \exp(-S_{eff}[\mathbf{n}_0])$ and hence a generalized Wilsonian RG transformation with all coupling constants running with λ . If we e.g. apply the same prescription to the 2D $O(N)$ σ model then in the one-loop calculation of Ref. [14] λ effectively acts as a mass squared for Goldstone modes, leading thus to the charge renormalization $\sim \ln(\Lambda/\sqrt{\lambda})$ (Λ is the ultraviolet cutoff). Hence changing λ is indeed similar to changing the scale. We now argue, that beyond the perturbation theory λ also may be viewed as a scale parameter.

For λ large enough only paths close to $\mathbf{n}_0(\tau)$ contribute to the path integral (2). But for the particle on the ring it is possible that a given path $\mathbf{n}(\tau)$ is close to $\mathbf{n}_0(\tau)$ for the most of the time, but suddenly makes a fast complete rotation around the ring in time τ_0 . For such instanton-like paths the weight factor (3) behaves as $w \sim \exp(-\text{const} \times \lambda \tau_0)$, so that “instantons” with size $\tau_0 > 1/\lambda$ are strongly suppressed (very fast rotations with $\tau_0 \ll m$ are suppressed by the kinetic term in Eq. (1)). Then with decreasing λ more and more instanton-like paths of larger scale contribute to the integral (2). Clearly, this is exactly what a physicist usually expects from the RG transformation in theories with instantons.

If we combine the action (1) with the exponential from (3) then the resulting action in (2) (up to a constant)

$$S[\mathbf{n}] = S_0[\mathbf{n}] + \lambda \int_0^\beta \mathbf{n}(\tau) \mathbf{n}_0(\tau) d\tau \quad (4)$$

describes the particle on the ring in time dependent electric field $\lambda \mathbf{n}_0(\tau)$. For slowly varying $\mathbf{n}_0(\tau)$ at zero temperature one can treat this problem in the adiabatic approximation. Then, if the electric field makes one complete revolution, the ground state will turn back to itself up to a phase factor (Berry phase [15]) which we denote by $\exp(i\theta')$. If we introduce polar angles ϕ and ϕ_0 instead of the vectors \mathbf{n} and \mathbf{n}_0 then the corresponding Hamiltonian may be written as

$$H = \frac{1}{2m} \left(-i \frac{\partial}{\partial \phi} - \frac{\theta}{2\pi} \right)^2 + \lambda \cos(\phi - \phi_0(t)) \quad (5)$$

Let $\psi_0(\phi) = \psi_0(\phi - \phi_0)$ be the instantaneous ground state wavefunction for the Hamiltonian (5) with the energy E_0 , which obviously does not depend on ϕ_0 . Then the Berry phase for the adiabatic change of ϕ_0 from zero to 2π is given by [15]

$$\theta' = i \int_0^{2\pi} d\phi_0 \langle \psi_0 | \frac{\partial}{\partial \phi_0} | \psi_0 \rangle \quad (6)$$

Since ψ_0 depends only on the difference $\phi - \phi_0$ we have

$$\begin{aligned} \langle \psi_0 | \frac{\partial}{\partial \phi_0} | \psi_0 \rangle &= -\langle \psi_0 | \frac{\partial}{\partial \phi} | \psi_0 \rangle = \\ &= -\langle \psi_0 | \left(\frac{\partial}{\partial \phi} - i \frac{\theta}{2\pi} \right) | \psi_0 \rangle - i \frac{\theta}{2\pi} \end{aligned} \quad (7)$$

The first term on the r.h.s. of Eq. (7) is proportional to the average of the derivative $\partial H / \partial \theta$ and hence

$$\theta' = \theta - 4\pi^2 m \frac{\partial E_0}{\partial \theta} \quad (8)$$

The nontrivial Berry phase, different from θ , means that the coarse grained continuously measured trajectory sees a “renormalized” magnetic field, as was discussed in [9], due to unobservable fast instanton-like rotations. This implies that for slowly varying \mathbf{n}_0 we should have

$$\begin{aligned} U[\mathbf{n}_0] \sim \exp\{ &-i \frac{\theta'}{2\pi} \int_0^\beta \epsilon_{ab} \dot{n}_0^a(\tau) \dot{n}_0^b(\tau) d\tau + \\ &+ \frac{m'}{2} \int_0^\beta \dot{\mathbf{n}}_0^2(\tau) d\tau + \dots \}, \end{aligned} \quad (9)$$

where dots indicate terms with higher derivatives of \mathbf{n}_0 and higher powers of $\dot{\mathbf{n}}_0$ and the renormalized mass will be determined below.

Similar origin of topological terms from a corresponding Berry phase was discussed in detail in Refs. [11] where fermions were coupled to the background vector field in various space-time dimensions (fermionic σ -models). Then integration over fermions results in Eq. (9) for planar vector \mathbf{n}_0 with θ' , m' dependent on the coupling constants. Here the fast mode which is integrated out is also the planar vector, so that it is more natural to speak of the θ renormalization rather than of the induced topological term.

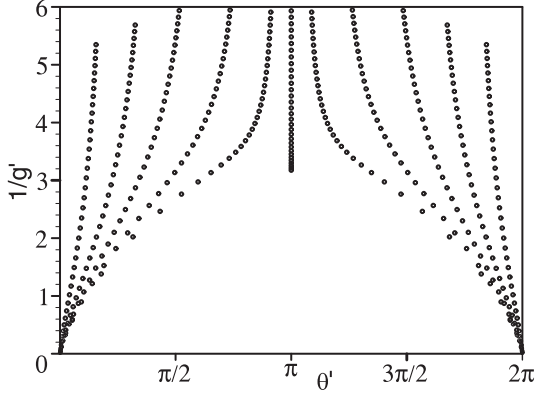


FIG. 1: Renormalized parameters $1/g' = \sqrt{m'\lambda}$ and θ' from Eqs. (8), (12) for different values of initial θ . λ decreases from top to bottom.

There exists a simple heuristic way to derive the expansion of Eq. (9). Consider a reference frame rotating with an angular frequency $\omega = \dot{\phi}_0$, which is assumed to be small and almost constant. In this frame \mathbf{n}_0 is constant, but an additional magnetic field $2m\omega$ is present according to the Larmor's theorem. Hence the hamiltonian H' in the rotating frame should be taken at the shifted value of the vacuum angle $\theta + 2m\pi\omega$, or more precisely,

$$H' = H + i\omega \frac{\partial}{\partial \phi} = H(\theta + 2m\pi\omega) - \frac{\theta}{2\pi}\omega - \frac{m}{2}\omega^2 \quad (10)$$

(see e.g. [16]), where the last term is the centrifugal potential (for the thin ring of unit radius) and the second one is due to the presence of the magnetic flux θ . Then if the particle is in its ground state the effective action (after Wick rotation $t \rightarrow -i\tau$ and expansion in powers of $\dot{\phi}_0$) may be written as

$$S_{eff} \simeq \int_0^\beta d\tau \left[\frac{m}{2} \dot{\phi}_0^2 - i \frac{\theta}{2\pi} \dot{\phi}_0 + E_0(\theta + 2m\pi\dot{\phi}_0) \right] = \text{const} + \int_0^\beta d\tau \left[\frac{m'}{2} \dot{\phi}_0^2 - i \frac{\theta'}{2\pi} \dot{\phi}_0 + \dots \right], \quad (11)$$

where θ' is given by the previously derived formula (8) and

$$m' = m - 4m^2\pi^2 \frac{\partial^2 E_0}{\partial \theta^2} \quad (12)$$

Clearly, this is the same action as in Eq. (9). Formulas (8) and (12) look very similar to the RG equations derived in [9]. Note, that they are independent of the specific form of the coupling between \mathbf{n} and \mathbf{n}_0 —all details are hidden in the ground state energy $E_0(\theta)$.

For large λ , when the effective electric field is strong, the θ dependence of E_0 is suppressed and $\theta' \simeq \theta$. In this case E_0 depends on θ only through instantons, as discussed in detail in [17], and

$$E_0(\theta) \simeq \text{const} - 2\sqrt{S_0} K e^{-S_0} \cos \theta \quad (13)$$

where $S_0(\lambda) \sim \sqrt{m\lambda}$ is the classical instanton action and $K = K(\lambda)$ results from the ratio of determinants [17]. Then in terms of dimensionless “coupling constants” $g = 1/\sqrt{m\lambda}$ and $g' = 1/\sqrt{m'\lambda}$ we finally have at $g \rightarrow 0$

$$\begin{aligned} \theta' &\simeq \theta - D(g)e^{-c/g} \sin \theta, \\ \frac{1}{g'^2} &\simeq \frac{1}{g^2} - \frac{1}{g^2} D(g)e^{-c/g} \cos \theta \end{aligned} \quad (14)$$

where c is some numerical constant and $D(g) = 8\pi^2 m K \sqrt{S_0}$. These equations are qualitatively similar to θ and charge renormalization due to instantons in QCD and σ models [2, 3].

If, on the other hand, λ tends to zero, then for the free motion on the ring $E_0 = (1/2m)(\theta/2\pi)^2$ for $\theta < \pi$, $E_0 = (1/2m)(\theta/2\pi - 1)^2$ for $\theta > \pi$ and Eqs. (8), (12) imply that $m' \rightarrow 0$ while $\theta' \rightarrow 0$, $\theta < \pi$ and $\theta' \rightarrow 2\pi$, $\theta > \pi$. These results are almost obvious, because at $\lambda = 0$ the slow field \mathbf{n}_0 is no longer coupled to \mathbf{n} .

In the close vicinity of the point $\theta = \pi$ the situation is more complicated. At $\lambda = 0$ the ground state is degenerate, but the degeneracy is lifted by arbitrarily small external potential. At small λ the energy gap may be expressed as $\delta E = a\sqrt{\lambda^2 + b(\theta - \pi)^2}$, where a and b are some numerical constants, and after expanding in $(\theta - \pi)$ near the maximum of $E_0(\theta)$ at $\theta = \pi$ we have

$$E_0(\theta) \simeq \text{const} - \frac{\alpha}{2\lambda}(\theta - \pi)^2, \quad (15)$$

where $\alpha = ab$. Hence from Eq. (12) $m' \rightarrow 4m^2\pi^2\alpha/\lambda$ at $\lambda \rightarrow 0$ and

$$1/g' = \sqrt{m'\lambda} \rightarrow 2m\pi\sqrt{\alpha} = \text{const}, \quad \theta = \pi \quad (16)$$

Thus for $\theta = \pi$ the coupling constant g' tends to a fixed value as $\lambda \rightarrow 0$. This is a kind of quantum mechanical anomaly (similar to “rotational anomaly” of Ref. [16]), since strictly at $\lambda = 0$ there is no interaction and m' should be equal to zero. Certainly, for very small λ when δE tends to zero near $\theta = \pi$ the adiabatic approximation used here becomes invalid.

Thus the dependence of m' and θ' on λ reproduces the main features of the famous QHE RG flow diagram. This can be seen from the Fig.1, where the evolution of the renormalized parameters is shown with λ decreasing from top to bottom for different initial values of the vacuum angle θ . The points in Fig.1 result from numerical calculation for a simplified model when the term $\lambda \cos \phi$ in Eq.(5) is replaced with $\lambda \delta(\phi)$ (qualitative features should not depend on the particular choice of the potential in Eq. (5)). Clearly, Fig.(1) is similar to the upper half of the QHE RG flow diagram with the unstable fixed point at $\theta = \pi$ and the ultimate flow of the renormalized vacuum angle to zero (mod 2π).

The quantum mechanical model discussed here enables, however, a transparent explanation of why the effective θ should vanish as $\lambda \rightarrow 0$. Let us add a kinetic term $(M/2)\dot{\mathbf{n}}_0^2$ for the field \mathbf{n}_0 with some large mass M ($M \gg m$ to ensure the adiabatic approximation) to the

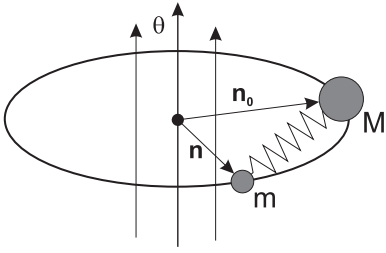


FIG. 2: Two particles with different masses ($M \gg m$) interacting via the harmonic potential on the ring with magnetic flux θ .

Lagrangian of Eq. (1). Then the resulting action with $w[\mathbf{n}, \mathbf{n}_0]$ from Eq. (3) taken into account describes two particles with masses m and M interacting via the harmonic potential, as shown in Fig.2. Note, that initially only the light particle interacts with the magnetic flux θ . One can say that the light particle is charged with, say, unit charge, while the heavy one is neutral.

Now, if λ , which determines the interparticle interaction strength, is high enough, two particles form a tightly bound pair or an “atom”, exactly with unit total charge. Mathematically this means, that the topological term for the field \mathbf{n}_0 is induced with $\theta' \simeq \theta$ due to the condensation of charge near the point \mathbf{n}_0 . When λ decreases, the bound state gets more loose. When the size of the bound state is of the order of the ring’s radius, rotations of the light particle are allowed (“instantons”) and its charge is spread along the ring. So the effective charge of the heavy particle reduces, which is seen in the formalism as the magnetic flux θ renormalization.

In summary, we demonstrate how the θ renormalization may appear in quantum mechanics of a particle,

moving around a thin ring threaded by a magnetic flux θ . Renormalized θ is a coefficient in the effective action for the slow variable $\mathbf{n}_0(\tau)$, which has the meaning of the coarse grained outcome of the measurement of the particle’s position. That is, if the position is measured with finite accuracy, the observed flux, equal to the Berry phase associated with the adiabatic rotation of \mathbf{n}_0 , will be smaller, than the true one. Formally this slow variable may be viewed as an additional degree of freedom, representing a second (heavy) particle on the ring, coupled to the first one with the harmonic force. Then renormalization of the flux θ may be also understood as arising from the change of the effective charge of the heavy particle when the interaction is changed.

This example shows, that while the renormalization of the vacuum angle is definitely a generic property of a system with instanton-like fluctuations (and the resulting RG flow is not particularly sensitive to the way the RG transformation is defined) it does not necessarily mean that observables are independent of θ , but is revealed, when the system is being measured or coupled to some additional slow variable. This mechanism, leading to small θ in effective low energy theory, looks physically different from the direct screening of θ , as e.g. in the case when the axion field is added, but it is still not clear whether it has any significance in QCD.

Acknowledgments

The author is grateful to V. Losyakov, A. Marshakov and especially to A. Morozov for valuable discussions. The work was supported in part by the RFBR grants No 06-02-17459 and No 07-02-01161.

-
- [1] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, New York, 1987).
 - [2] V.G. Knizhnik and A.Yu. Morozov, Pis'ma v ZhETF **39**, 202 (1984). [JETP Lett. **39**, 240 (1984)], H. Levine, and S. Libby, Phys. Lett. B **150**, 182 (1985).
 - [3] H. Levine, S. Libby, and A.M.M. Pruisken, Nucl. Phys. **B240** [FS12], 30, 49, 71 (1984), A.M.M. Pruisken, Nucl. Phys. **B290**, 61 (1987).
 - [4] J.I. Latorre and C.A. Lütken, Phys. Lett. B **421**, 217 (1998), A.M.M. Pruisken, M.A. Baranov, and M. Voropaev, cond-mat/0101003, L. Campos Venuti, C. Degli Esposti Boschi, E. Ercolessi, F. Ortolani, G. Morandi, S. Pasini, and M. Roncaglia, J. Stat. Mech. L02004 (2005), A.M.M. Pruisken, R. Shankar, and N. Surendran, Phys. Rev. B **72**, 035329 (2005), A.M.M. Pruisken and I.S. Burmistrov, Ann. of Phys. (N.Y.) **316**, 285 (2005).
 - [5] For a recent review see G. Gabadadze and M. Shifman, Int. J. Mod. Phys. A **17**, 3689 (2002).
 - [6] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979), M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B166**, 493 (1980).
 - [7] A.M.M. Pruisken, in *The Quantum Hall Effect*, eds. R.E. Prange and S. Girvin (Springer, 1990), A.M.M. Pruisken and I.S. Burmistrov, Ann. Phys. **322**, 1265 (2007).
 - [8] S.A. Bulgadaev, Pis'ma v ZhETF **83**, 659 (2006), cond-mat/0605360, I.S. Burmistrov and A.M.M. Pruisken, cond-mat/0702400.
 - [9] S.M. Apenko, Phys. Rev. B **74**, 193311 (2007).
 - [10] D.M. Ceperley, Rev. Mod. Phys. **67**, 279 (1995).
 - [11] M. Stone, Phys. Rev. D **33**, 1191 (1986), A.G. Abanov and P.B. Wigmann, Nucl. Phys. **B570**, 685 (2000).
 - [12] S.M. Apenko, Phys. Lett. A **142**, 277 (1989), G. Schön and A.D. Zaikin Phys. Rep. **198**, 237 (1990).
 - [13] R.P. Feynman, Rev. Mod. Phys. **20**, 367 (1948), M.B. Mensky, *Continuous Quantum Measurements and Path Integrals* (IOP Publishing, 1993).
 - [14] A.M. Polyakov, Phys. Lett. B **59**, 79 (1975).
 - [15] M. Berry, Proc. Roy. Soc. London, **A392**, 45 (1984).
 - [16] R. Merlin, Phys. Lett. A **18**, 421 (1993).
 - [17] R. Rajaraman, *Solitons and Instantons* (North Holland, 1982).